

FOLDING QUIVERS AND NUMERICAL STABILITY CONDITIONS

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ABSTRACT. We generalize Deng-Du's folding argument, for the bounded derived category $\mathcal{D}^b(Q)$ of an acyclic quiver Q , to the finite dimensional derived category $\mathcal{D}(\Gamma Q)$ of the Ginzburg algebra ΓQ associated to Q . We show that the F-stable category of $\mathcal{D}(\Gamma Q)$ is equivalent to the finite dimensional derived category $\mathcal{D}(\Gamma \mathbb{S})$ of the Ginzburg algebra $\Gamma \mathbb{S}$ associated to the specie \mathbb{S} , which is folded from Q . Then we show that, if (Q, \mathbb{S}) is of Dynkin type, the principal component $\text{Stab}_0 \mathcal{D}(\Gamma \mathbb{S})$ of the space of the stability conditions of $\mathcal{D}(\Gamma \mathbb{S})$ is canonically isomorphic to the principal component $\text{Stab}_0^F \mathcal{D}(\Gamma Q)$ of the space of F-stable stability conditions of $\mathcal{D}(\Gamma Q)$. As an application, we show that, if (Q, \mathbb{S}) is of type (A_3, B_2) or (D_4, G_2) , the space $\text{Stab}^N \mathcal{D}(\Gamma Q)$ of numerical stability conditions in $\text{Stab}_0 \mathcal{D}(\Gamma Q)$, consists of $\text{Br } \Gamma Q / \text{Br } \Gamma \mathbb{S}$ many connected components, each of which is isomorphic to $\text{Stab}_0 \mathcal{D}(\Gamma \mathbb{S}) \cong \text{Stab}_0^F \mathcal{D}(\Gamma Q)$.

Key words: Calabi-Yau category; Frobenius morphism; folding; t-structure; numerical stability conditions

INTRODUCTION

Bridgeland [3] defined the notion of a stability condition on a triangulated category, trying to understand D-branes in string theory from a mathematical point of view. We aim to study the spaces of stability conditions arising from representation theory. Namely, the bounded/finite dimensional derived categories for quivers and species.

One of our main techniques is folding, which is well-known in studying non-simply laced Dynkin diagram. In particular, folding the bounded derived category $\mathcal{D}(Q)$ of a quiver Q was studied by Deng-Du ([1], [2]). The key observation is that an automorphism on the quiver Q will induce a Frobenius morphism on the path algebra $\mathbf{k}Q$ and a Frobenius morphism (which is an auto-equivalence) on the category $\mathcal{D}(Q)$. Then the F-stable category of $\mathcal{D}(Q)$ is derived equivalent to the bounded derived category of a specie \mathbb{S} , which is obtained by folding the quiver Q . Our first aim is (Proposition 2.1) to generalize this result to the finite dimensional derived category $\mathcal{D}(\Gamma Q)$ of the Ginzburg algebra ΓQ associated to Q . Namely, the F-stable category of $\mathcal{D}(\Gamma Q)$ is derived equivalent to the finite dimensional derived category $\mathcal{D}(\Gamma \mathbb{S})$ of the Ginzburg algebra $\Gamma \mathbb{S}$ associated to \mathbb{S} . We will then show (Corollary 4.3) that, if (Q, \mathbb{S}) is of Dynkin type, the principal component $\text{Stab}_0 \mathcal{D}(\Gamma \mathbb{S})$ of the space of the stability conditions of $\mathcal{D}(\Gamma \mathbb{S})$ is canonically isomorphic to the principal component $\text{Stab}_0^F \mathcal{D}(\Gamma Q)$ of the space of F-stable stability conditions of $\mathcal{D}(\Gamma Q)$. As an application, we show our main result (Theorem 5.1) that, if (Q, \mathbb{S}) is of type (A_3, B_2) or (D_4, G_2) , the space $\text{Stab}^N \mathcal{D}(\Gamma Q)$ of

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numerical stability conditions in $\text{Stab}_0 \mathcal{D}(\Gamma Q)$, consists of $\text{Br } \Gamma Q / \text{Br } \Gamma \mathbb{S}$ many connected components, each of which is isomorphic to $\text{Stab}_0 \mathcal{D}(\Gamma \mathbb{S}) \cong \text{Stab}_0^F \mathcal{D}(\Gamma Q)$.

This paper was motivated by a talk with Tom Sutherland and Alastair King. Sutherland studies a list of quivers (known as Painlevé quivers) in his PhD work [14], whose corresponding spaces of numerical stability conditions are related to elliptic surface. Note that all those quivers are foldable except one. Moreover, the space of the numerical stability space for a quiver Q is related to the cluster algebra, whose type is the corresponding specie \mathbb{S} folded from Q (cf. [11]).

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1. PRELIMINARIES

1.1. Notations. Let \mathbb{F}_q be a finite field with q elements and $\mathbf{k} = \overline{\mathbb{F}_q}$ be its algebraic closure with field automorphism $\varsigma = \varsigma_q$ ($k \mapsto k^q$). In this paper, all categories will be implicitly assumed to be \mathbb{F}_q -linear or \mathbf{k} -linear.

Throughout this paper, we fix the following notations:

- Let (Q, σ) be an acyclic quiver with admissible automorphism and Frobenius morphism F as in (A.1); let $\mathbb{S} = (\mathbb{S}, L_{\mathbf{i}}, X_{\mathbf{a}})$ be the folded specie corresponding to (Q, σ) , see Appendix A for details.
- Denote by $\mathbf{k}Q = \mathbf{k}Q_0 \langle a \mid a \in Q_1 \rangle$ and $F_q \mathbb{S} = L \langle X_{\mathbf{a}} \mid \mathbf{a} \in \mathbb{S}_1 \rangle$ the path algebras of Q and \mathbb{S} , respectively, and let

$$\mathcal{H}_Q = \text{mod } \mathbf{k}Q, \quad \mathcal{H}_{\mathbb{S}} = \text{mod } \mathbb{F}_q \mathbb{S},$$

and their bounded derived categories be $\mathcal{D}(Q)$ and $\mathcal{D}(\mathbb{S})$ respectively.

- Denote by $\text{Sim } \mathcal{A}$ a complete set of simples of an abelian category \mathcal{A} . Let

$$\text{Sim } \mathcal{H}_Q = \{S_i \mid i \in Q_0\}, \quad \text{Sim } \mathcal{H}_{\mathbb{S}} = \{S_{\mathbf{i}} \mid \mathbf{i} \in \mathbb{S}_0\}.$$

- Let ΓQ and $\Gamma \mathbb{S}$ be the Ginzburg algebra of Q and \mathbb{S} , respectively, and denote by $\mathcal{D}(\Gamma Q)$ and $\mathcal{D}(\Gamma \mathbb{S})$ the derived categories of finite dimensional dg modules for ΓQ and $\Gamma \mathbb{S}$. See Appendix B for details.
- Note that $\mathcal{D}(\Gamma Q)$ and $\mathcal{D}(\Gamma \mathbb{S})$ admit standard hearts

$$\mathcal{G}_Q \cong \mathcal{H}_Q \quad \text{and} \quad \mathcal{G}_{\mathbb{S}} \cong \mathcal{H}_{\mathbb{S}}, \tag{1.1}$$

respectively (cf. [6]), and denote their simples by

$$\text{Sim } \mathcal{G}_Q = \{T_i \mid i \in Q_0\}, \quad \text{Sim } \mathcal{G}_{\mathbb{S}} = \{T_{\mathbf{i}} \mid \mathbf{i} \in \mathbb{S}_0\}.$$

- Denote by $\text{Br } \Gamma Q$ and $\text{Br } \Gamma \mathbb{S}$, respectively, the *Seidel-Thomas braid group* generated by the twist functors as in Appendix B.

Example 1.1. When Q is of Dynkin type, all possible admissible automorphism σ and the corresponding specie \mathbb{S} are

- 1°. Q is of type D_{n+1} and \mathbb{S} is of type B_n while σ exchanges the bullets.

$$D_{n+1} \quad \circ - \circ - \cdots - \circ \begin{array}{l} \nearrow \bullet \\ \searrow \bullet \end{array} \quad \Bigg)_{\sigma} \quad B_n \quad \underset{1}{\circ} - \underset{1}{\circ} - \cdots - \underset{1}{\circ} = \underset{2}{\bullet}$$

2°. Q is of type A_{2n+1} and \mathbb{S} is of type C_n while σ exchanges the bullets in the same collum.

$$A_{2n+1} \quad \sigma \left(\begin{array}{c} \bullet - \bullet - \dots - \bullet \\ \bullet - \bullet - \dots - \bullet \end{array} \right) \circ \quad C_n \quad \begin{array}{c} \bullet - \bullet - \dots - \bullet = \circ \\ 2 \quad 2 \quad \quad \quad 2 \quad 1 \end{array}$$

3°. Q is of type E_6 and \mathbb{S} is of type F_4 while σ exchanges the bullets in the same collum.

$$E_6 \quad \circ - \circ \left(\begin{array}{c} \bullet - \bullet \\ \bullet - \bullet \end{array} \right) \sigma \quad F_4 \quad \begin{array}{c} \circ - \circ = \bullet - \bullet \\ 1 \quad 1 \quad 2 \quad 2 \end{array}$$

4°. Q is of type D_4 and \mathbb{S} is of type G_2 while σ permutes the three bullets.

$$D_4 \quad \circ \left(\begin{array}{c} \bullet \\ \bullet \\ \bullet \end{array} \right) \sigma \quad G_2 \quad \begin{array}{c} \circ \equiv \bullet \\ 1 \quad 3 \end{array}$$

Note that the orientation of arrows involving the circles need to be consistent with σ and the label on vertex \mathbf{i} in \mathbb{S} are $|\mathbf{i}|$.

1.2. Tilting. A similar notion to a *torsion pair* in an abelian category is a t-structure on a triangulated category \mathcal{D} . A *t-structure* is the torsion part of some torsion pair in \mathcal{D} that is closed under positive shifts. When the t-structure is *bounded* (which we will always assume in this paper), it uniquely corresponds to a subcategory of \mathcal{D} , known as its *heart*. There is a natural partial order of hearts, that $\mathcal{H}_1 \leq \mathcal{H}_2$ if and only if $\mathcal{P}_1 \supset \mathcal{P}_2$, where \mathcal{P}_i is the corresponding t-structure of \mathcal{H}_i .

By [5], for any heart \mathcal{H} (in a triangulated category) with torsion pair $\langle \mathcal{F}, \mathcal{T} \rangle$, there exists the following two hearts with torsion pairs

$$\mathcal{H}^\sharp = \langle \mathcal{T}, \mathcal{F}[1] \rangle, \quad \mathcal{H}^\flat = \langle \mathcal{T}[-1], \mathcal{F} \rangle.$$

We call \mathcal{H}^\sharp the *forward tilt* of \mathcal{H} with respect to the torsion pair $\langle \mathcal{F}, \mathcal{T} \rangle$, and \mathcal{H}^\flat the *backward tilt* of \mathcal{H} .

We say a forward tilt is *simple*, if the corresponding torsion free part is generated by a single rigid simple object S (as in Lemma C.1). We denote the new heart by \mathcal{H}_S^\sharp . Similarly, a backward tilt is simple if the corresponding torsion part is generated by such a simple and the new heart is denoted by \mathcal{H}_S^\flat .

The notion of simple titling leads to exchange graphs. Define the *total exchange graph* $\text{EG } \mathcal{D}$ of a triangulated category \mathcal{D} to be the oriented graph whose vertices are all hearts in \mathcal{D} and whose edges correspond to simple forward tiltings between them.

We have the following consequence of Proposition C.2.

Lemma 1.2. *Let \mathcal{H} be a heart in a triangulated category \mathcal{D} with rigid simples R_1, \dots, R_m such that $\text{Hom}^\bullet(R_i, R_j) = 0$ for any $1 \leq i, j \leq m$. Then there is a torsion pair $\langle \mathcal{F}, \mathcal{T} \rangle$ such that the forward tilt of \mathcal{H} with respect to which equals to \mathcal{H}_m , where*

$$\mathcal{H}_0 = \mathcal{H}, \quad \mathcal{H}_i = (\mathcal{H}_{i-1})_{R_{\rho(i)}}^\sharp$$

and ρ is any fixed permutation of m elements. Denote the tilt \mathcal{H}_m of \mathcal{H} by $\mathcal{H}_{R_1, \dots, R_m}^\sharp$.

Proof. Fix a permutation ρ . By repeatedly using Proposition C.2, we inductively deduce that any simple in \mathcal{H}_i , for $0 \leq i \leq m$, admits a filtration of triangles with factors in $\text{Sim } \mathcal{H} \cup \text{Sim } \mathcal{H}[1]$. Thus, its homology with respect to \mathcal{H} lives only in degree zero and one. By [9, Lemma 5.4], this means $\mathcal{H} \leq \mathcal{H}_i \leq \mathcal{H}[1]$. By [11, Lemma 2.9], we know that \mathcal{H}_i is the forward tilt of \mathcal{H} with respect to some torsion pair. Then, using formulae in Proposition C.2, a direct calculation shows that the tilt \mathcal{H}_m is independent of the choice of permutation, as required. \square

1.3. Stability conditions. A *stability condition* on \mathcal{D} consists of a heart \mathcal{H} and a *stability function* Z on \mathcal{H} with the *Harder-Narashimhan property*, which will be denoted by (\mathcal{H}, Z) . The function Z is known as the *central charge*, which is a group homomorphism from the Grothendieck group $K(\mathcal{H}) \cong K(\mathcal{D})$ to \mathbb{C} . Recall a crucial result due to Bridgeland.

Theorem 1.3. [3, Theorem 1.2] *All stability conditions on a triangulated category \mathcal{D} form a complex manifold, denoted by $\text{Stab}(\mathcal{D})$; each connected component of $\text{Stab}(\mathcal{D})$ is locally homeomorphic to a linear sub-manifold of $\text{Hom}_{\mathbb{Z}}(K(\mathcal{D}), \mathbb{C})$, sending a stability condition (\mathcal{H}, Z) to its central charge Z .*

Note that every finite heart \mathcal{H} , that is, a heart generating by finitely many simples, corresponds to a (complex, half closed and half open) n -cell

$$U(\mathcal{H}) \simeq H^n$$

inside $\text{Stab}(\mathcal{D})$, where

$$H := \{r \exp(i\pi\theta) \mid r \in \mathbb{R}_{>0}, 0 \leq \theta < 1\} \quad (1.2)$$

is the upper half plane of the complex plane \mathbb{C} . For more details, see [3] or [11, Section 2].

1.4. Frobenius morphisms. By [1] and [2], we have an algebra isomorphism $(\mathbf{k}Q)^F \cong \mathbb{F}_q \mathbb{S}$. Moreover, there is a Frobenius functor in $\text{Aut } \mathcal{D}(Q)$ induced by F , which also denote by F . For any object M in $\mathcal{D}(Q)$, denote by $p(M)$ the F -period of M , that is, the minimal positive integer m , such that $F^m(M) = M$. Note that, by [2], $p(M)$ is finite for any M and let

$$\widetilde{M} = \bigoplus_{j=1}^{p(M)} F^j(M). \quad (1.3)$$

We say M is an F -stable object if $p(M) = 1$, i.e. $F(M) = M$. We say a subcategory \mathcal{C} of $\mathcal{D}(Q)$ is F -stable if $F(\mathcal{C}) = \mathcal{C}$.

Let $\mathcal{D}(Q)^F$ be the F -stable category of $\mathcal{D}(Q)$, whose objects are F -stable objects of $\mathcal{D}(Q)$ and whose morphisms are the ones which commutes with F (see [2, Remark 5.5] for details). Then we have a derived equivalence ([2, Theorem 5.4])

$$\Phi: \mathcal{D}(\mathbb{S}) \cong \mathcal{D}(Q)^F \quad (1.4)$$

such that

$$\Phi(\mathcal{H}_{\mathbb{S}}) = \mathcal{H}_Q^F \quad \text{and} \quad \Phi(S_i) = \bigoplus_{i \in \mathbf{i}} S_i.$$

The second equation is equivalent to $\Phi(S_i) = \tilde{S}_i$ for any $i \in \mathbf{i}$. We will write $\tilde{S}_{\mathbf{i}}$ for $\Phi(S_{\mathbf{i}})$. Further, for any $X \in \text{Ind } \mathcal{D}(Q)^F$, there exists $M \in \text{Ind } \mathcal{D}(Q)$ such that $X = \tilde{M}$.

Similar to (A.1), we have the Frobenius morphism on ΓQ such that the restriction to $\mathbf{k}Q$ (i.e. the degree zero part of ΓQ) is exactly (A.1). This morphism also induces a Frobenius functor, still denoted by F , on $\mathcal{D}(\Gamma Q)$, which is an auto-equivalence. Denote the F-stable category of $\mathcal{D}(\Gamma Q)$ as $\mathcal{D}(Q)^F$ by $\mathcal{D}(\Gamma Q)^F$. Then we have the following diagram (cf. Appendix C)

$$\begin{array}{ccccc} \mathcal{D}(\mathbb{S}) & \xrightarrow[\cong]{\Phi} & \mathcal{D}(Q)^F & \hookrightarrow & \mathcal{D}(Q) \\ \mathcal{L}_{\mathbb{S}} \downarrow & & & & \downarrow \mathcal{L}_Q \\ \mathcal{D}(\Gamma \mathbb{S}) & \dashrightarrow^? & \mathcal{D}(\Gamma Q)^F & \dashrightarrow^? & \mathcal{D}(\Gamma Q) \end{array} \quad (1.5)$$

2. FOLDING CALABI-YAU CATEGORY

In this section, we aim to complete (1.5).

Proposition 2.1. *There is a faithful functor $\Theta: \mathcal{D}(\Gamma \mathbb{S}) \rightarrow \mathcal{D}(\Gamma Q)$, sending $T_{\mathbf{i}}$ to $\tilde{T}_{\mathbf{i}} = \bigoplus_{i \in \mathbf{i}} T_i$ and inducing a derived equivalence*

$$\Theta: \mathcal{D}(\Gamma \mathbb{S}) \cong \mathcal{D}(\Gamma Q)^F. \quad (2.1)$$

Proof. By (1.1), we know that \tilde{T}_i as in (1.3) is well-defined for $i \in Q$ and

$$\tilde{T}_{\mathbf{i}} := \tilde{T}_i \in \mathcal{D}(\Gamma Q)^F$$

for any $i \in \mathbf{i}$. By the properties of L-immersions (in Proposition C.3) and (1.4), we can calculate that

$$\text{End}_{\mathcal{D}(\Gamma Q)^F}^{\bullet} \left(\bigoplus_{\mathbf{i} \in \mathbb{S}_0} \tilde{T}_{\mathbf{i}} \right) \cong \text{End}_{\mathcal{D}(\Gamma \mathbb{S})}^{\bullet} \left(\bigoplus_{\mathbf{i} \in \mathbb{S}_0} T_{\mathbf{i}} \right).$$

which implies that the existence of the required faithful functor Θ . Moreover, we only need to show that $\{\tilde{T}_{\mathbf{i}}\}_{\mathbf{i} \in \mathbb{S}_0}$ generates $\mathcal{D}(\Gamma Q)^F$ for (2.1).

First, $\{\tilde{T}_{\mathbf{i}}\}_{\mathbf{i} \in \mathbb{S}_0}$ generates \mathcal{G}_Q^F as $\{\tilde{S}_{\mathbf{i}}\}_{\mathbf{i} \in \mathbb{S}_0}$ generates \mathcal{H}_Q^F by (1.1). Second, any $M \in \mathcal{D}(\Gamma Q)^F$ admits a canonical filtration

$$\begin{array}{ccccccc} 0 = M_0 & \longrightarrow & M_1 & \longrightarrow & \cdots & \longrightarrow & M_{m-1} & \longrightarrow & M_m = M \\ & & \nwarrow & & & & \nwarrow & & \\ & & H_1[k_1] & & & & H_m[k_m] & & \end{array} \quad (2.2)$$

where $H_i \in \mathcal{G}_Q$ and $k_1 > \dots > k_m$ are integers. Since M is F-stable and the filtration is unique, we deduce that each *homology* H_i of M , with respect to \mathcal{G}_Q , is F-stable. Hence, any of these homologies is generated by $\{\tilde{T}_{\mathbf{i}}\}_{\mathbf{i} \in \mathbb{S}_0}$, and so is M as required. \square

Thus it is straightforward to see that (1.5) is completed to the commutative diagram

$$\begin{array}{ccccc}
 \mathcal{D}(\mathbb{S}) & \xrightarrow[\cong]{\Phi} & \mathcal{D}(Q)^F & \hookrightarrow & \mathcal{D}(Q) \\
 \mathcal{L}_{\mathbb{S}} \downarrow & & \downarrow \mathcal{L}_Q & & \downarrow \mathcal{L}_Q \\
 \mathcal{D}(\Gamma\mathbb{S}) & \xrightarrow[\cong]{\Theta} & \mathcal{D}(\Gamma Q)^F & \hookrightarrow & \mathcal{D}(\Gamma Q)
 \end{array} \tag{2.3}$$

3. F-STABLE FOR BOUNDED DERIVED CATEGORIES

For a F-stable heart \mathcal{H} in $\mathcal{D}(Q)$, define its F-stabilization to be the abelian category \mathcal{H}^F in $\mathcal{D}(Q)^F$, consisting of objects $\{\widetilde{M} \mid M \in \mathcal{H}\}$, where $\widetilde{?}$ is as in (1.3).

Lemma 3.1. *If \mathcal{H} is a F-stable heart in $\mathcal{D}(Q)$, then its F-stabilization \mathcal{H}^F is a heart in $\mathcal{D}(Q)^F$.*

Proof. We use the criterion in [3, Lemma 3.2] for the definition of hearts, namely, an abelian category \mathcal{A} in a triangulated category \mathcal{D} is a heart if and only if it satisfies the following conditions

- $\text{Hom}_{\mathcal{D}}(A[a], B[b]) = 0$ for any $A, B \in \mathcal{A}$ and any integer $a > b$.
- There is a canonical filtration (2.2) for any M in \mathcal{D} with $H_i \in \mathcal{A}$ and $k_1 > \dots > k_m$ are integers.

For any X and Y in \mathcal{H}^F , there exists M and L in \mathcal{H} such that $X = \widetilde{M}$ and $Y = \widetilde{L}$ as in (1.3). Since $\text{Hom}_{\mathcal{D}(Q)}(A[a], B[b]) = 0$ for any $A, B \in \mathcal{H}$ and any integer $a > b$, we have $\text{Hom}_{\mathcal{D}(Q)^F}(X[a], Y[b]) = 0$.

Further, there is a canonical filtration (2.2) of M with $H_i \in \mathcal{H}$ and $k_1 > \dots > k_m$ are integers. Since \mathcal{H} is F-stable, $F^j(H_i)$ is also in \mathcal{H} which implies the canonical filtration of $F^j(M)$ has factors $F^j(H_1)[k_1], \dots, F^j(H_m)[k_m]$. By direct summing the triangles in the canonical filtrations of $F^j(M)$, for $j = 1, \dots, p(M)$, we obtain a filtration of \widetilde{M} in $\mathcal{D}(Q)$, with factors

$$\bigoplus_{j=1}^{p(M)} F^j(H_1)[k_1], \dots, \bigoplus_{j=1}^{p(M)} F^j(H_m)[k_m].$$

To see that this induces the canonical filtration of $X = \widetilde{M}$ in $\mathcal{D}(Q)^F$ (under Φ), we only need to show that $\bigoplus_{j=1}^{p(M)} F^j(H_i)[k_i]$ is F-stable, or equivalently, $F^{p(M)}(H_i) = H_i$. This follows by comparing the canonical filtrations of $F^{p(M)}(M)$ and M , noticing that the canonical filtration is unique. Therefore, \mathcal{H}^F is a heart in $\mathcal{D}(Q)^F$. \square

Note that the $\text{Aut } \mathcal{D}$ acts on the Grothendieck group $K(\mathcal{D})$, as well as $\text{Stab } \mathcal{D}$ in a natural way. Denote by $K^F(\mathcal{D}(Q))$ the subgroup in $K(\mathcal{D}(Q))$ consisting of F-stable elements. Then we have a canonical isomorphism $K^F(\mathcal{D}(Q)) \cong K(\mathcal{D}(Q)^F)$. Further, denote by $\text{Stab}^F \mathcal{D}(Q)$ the subspace in $\text{Stab } \mathcal{D}(Q)$ consisting of F-stable stability conditions.

An immediate corollary for stability conditions is as follow.

Corollary 3.2. *There is a canonical inclusion*

$$\iota_Q : \text{Stab}^F \mathcal{D}(Q) \rightarrow \text{Stab } \mathcal{D}(Q)^F \tag{3.1}$$

sending a stability condition (\mathcal{H}, Z) to (\mathcal{H}^F, Z^F) , where $Z^F(\widetilde{M}) = Z(M)$, for any $M \in \mathcal{D}(Q)$. Moreover, we have the following commutative diagram

$$\begin{array}{ccc} \mathrm{Stab}^F \mathcal{D}(Q) & \xrightarrow{\iota_Q} & \mathrm{Stab} \mathcal{D}(Q)^F \\ \downarrow & & \downarrow \\ \mathrm{Hom}(\mathrm{K}^F(\mathcal{D}(Q)), \mathbb{C}) & \xrightarrow{\cong} & \mathrm{Hom}(\mathrm{K}(\mathcal{D}(Q)^F), \mathbb{C}), \end{array} \quad (3.2)$$

where the vertical maps are the canonical local homeomorphism in Theorem 1.3, sending a stability condition to its central charge.

Proof. Clearly, the map ι_Q is a locally well-defined and [11, (3.?)] implies that we can glue them together as required. \square

We think that the inverse of Lemma 3.1 is also true, but we only (need and) prove a partial result. Denote by $\mathrm{EG}^\circ \mathcal{D}(Q)$ and $\mathrm{EG}^\circ \mathcal{D}(\mathbb{S})$ the principal component of the exchange graph $\mathrm{EG} \mathcal{D}(Q)$ and $\mathrm{EG} \mathcal{D}(\mathbb{S})$, respectively. Note the derived equivalence (1.4) induces an isomorphism $\Phi: \mathrm{EG}^\circ \mathcal{D}(\mathbb{S}) \rightarrow \mathrm{EG}^\circ \mathcal{D}(Q)^F$.

Proposition 3.3. *Any heart in $\Phi(\mathrm{EG}^\circ \mathcal{D}(\mathbb{S}))$ is the F -stabilization of some heart in $\mathrm{EG}^\circ \mathcal{D}(Q)$. Moreover, if $\Phi(\mathcal{H}) = \widetilde{\mathcal{H}}^F$ for hearts $\mathcal{H} \in \mathrm{EG}^\circ(\mathcal{D}(\mathbb{S}))$ and $\widetilde{\mathcal{H}} \in \mathrm{EG}^\circ \mathcal{D}(Q)$, we have the following.*

- 1°. $\mathrm{Sim} \mathcal{H}$ and $\mathrm{Sim} \widetilde{\mathcal{H}}$ can be written as $\{R_{\mathbf{i}}\}_{\mathbf{i} \in \mathbb{S}_0}$ and $\{R_i\}_{i \in Q_0}$, respectively, such that $\Phi(R_{\mathbf{i}}) = \bigoplus_{i \in \mathbf{i}} R_i$.
- 2°. For any $\mathbf{i} \in \mathbb{S}_0$ and $i_1, i_2 \in \mathbf{i}$, $\mathrm{Hom}^\bullet(R_{i_1}, R_{i_2}) = 0$.
- 3°. For any $\mathbf{i} \in \mathbb{S}_0$, we have

$$\Phi(\mathcal{H}^\sharp) = (\widetilde{\mathcal{H}}^\sharp)^F \quad \text{and} \quad \Phi(\mathcal{H}^\flat) = (\widetilde{\mathcal{H}}^\flat)^F, \quad (3.3)$$

where the tilts of \mathcal{H} are with respect to $R_{\mathbf{i}}$ and the tilts of $\widetilde{\mathcal{H}}$ are with respect to the set of simples $\{R_i\}_{i \in \mathbf{i}}$ in the sense of Lemma 1.2.

Proof. We use induction starting from the standard hearts $\Phi(\mathcal{H}_{\mathbb{S}}) = \mathcal{H}_Q^F$ satisfying 1° and 2°. We only need to show that, if $(\mathcal{H}, \widetilde{\mathcal{H}})$ satisfy $\Phi(\mathcal{H}) = \widetilde{\mathcal{H}}^F$, 1° and 2°, then they also satisfy 3° and hearts in (3.3) satisfy 1° and 2°.

Let $(\mathcal{H}, \widetilde{\mathcal{H}})$ satisfy $\Phi(\mathcal{H}) = \widetilde{\mathcal{H}}^F$, 1° and 2°. Consider a fixed $\mathbf{i} \in \mathbb{S}_0$. Suppose that the orbit \mathbf{i} contains vertices $1, \dots, |\mathbf{i}|$ in Q_0 . We claim that

$$(\widehat{\mathcal{H}}^F)_{\widetilde{R}_{\mathbf{i}}}^\sharp = \left(\widehat{\mathcal{H}}_{R_1, \dots, R_{|\mathbf{i}|}}^\sharp \right)^F, \quad (3.4)$$

where $\widetilde{R}_{\mathbf{i}} = \widetilde{R}_i = \bigoplus_{i \in \mathbf{i}} R_i$. Because of 2°, RHS of (3.4) is well-defined in the sense of Lemma 1.2. Since the simples determine a heart, (3.4) is equivalent to the equality between their sets of simples. Let $\mathbf{j} \in \mathbb{S}_0$ and contains vertices $1', \dots, |\mathbf{j}'|$ in Q_0 with corresponding simples $R_{j'} \in \mathrm{Sim} \mathcal{H}$. Let $\widetilde{R}_{\mathbf{j}} = \widetilde{R}_{j'} = \bigoplus_{j' \in \mathbf{j}} R_{j'}$. By formulae in

Proposition C.2, we only need to show

$$\psi_{\tilde{R}_i}^\#(\tilde{R}_j) = \bigoplus_{j=1'}^{|\mathbf{j}|'} \Psi(R_{j'}) \quad (3.5)$$

where $\Psi = \psi_{R_1}^\# \circ \dots \circ \psi_{R_{|\mathbf{i}|}}^\#$ and $\psi^\#$ is defined as in (C.4).

Let $d = \gcd(|\mathbf{i}|, |\mathbf{j}|)$ and $|\mathbf{i}| = sd$, $|\mathbf{j}| = td$ for some integer s, t . Without loss of generality, suppose that $F(R_k) = R_{k+1}$ and $F(R_{k'}) = R_{(k+1)'}$, where $R_k = R_{k+sd}$ and $R_{k'} = R_{(k+td)'}$. Further, suppose that

$$\mathrm{Ext}_{\mathcal{D}(Q)}^1(R_{k'}, R_0) = \mathbf{k}^{h_{k'}}$$

for $k = 1, \dots, d$. Then, by applying the Frobenius functor, we have

$$\mathrm{Ext}_{\mathcal{D}(Q)}^1(R_{j'}, R_i) = \mathbf{k}^{h_{j'} - i},$$

where $h_x = h_{x+d}$ for any $x \in \mathbb{Z}$. Using formula (C.4), a direct calculation shows that $\Psi(R_{j'})$ admit a filtration of triangles in $\mathcal{D}(Q)$, with factors

$$R_1^{h_{j-1}}, \dots, R_{sd}^{h_{j-sd}}, R_{j'},$$

for any $1 \leq j \leq td$. Noticing that $\mathrm{Hom}^\bullet(R_{i_1}, R_{i_2}) = 0$, we actually have triangles

$$R_{j'}[-1] \rightarrow \bigoplus_{i=1}^{sd} R_i^{h_{j-i}} \rightarrow \Psi(R_{j'}) \rightarrow R_{j'}.$$

Direct summing these triangles gives a triangle

$$\tilde{R}_j[-1] \xrightarrow{\alpha} \tilde{R}_i^{t \cdot h} \rightarrow \bigoplus_{j=1'}^{|\mathbf{j}|'} \Psi(R_{j'}) \rightarrow \tilde{R}_j, \quad (3.6)$$

in $\mathcal{D}(Q)$, where $h = \sum_{k=1}^d h_k$. By the definition of $\psi_{R_i}^\#$, α in (3.6) contains all maps in $\mathrm{Hom}_{\mathcal{D}(Q)}(\tilde{R}_j[-1], \tilde{R}_i^{t \cdot h})$.

On the other hand, we have

$$\mathrm{Ext}_{D(Q)^F}^1(\tilde{R}_j, \tilde{R}_i) = \mathbb{F}_{q^{t \cdot s \cdot d \cdot h}}, \quad \mathrm{End}_{D(Q)^F}(\tilde{R}_i, \tilde{R}_i) = \mathbb{F}_{q^{s \cdot d}}.$$

Then (C.4) gives a triangle

$$\tilde{R}_j[-1] \xrightarrow{\alpha'} \tilde{R}_i^{t \cdot h} \rightarrow \psi_{\tilde{R}_i}^\#(\tilde{R}_j) \rightarrow \tilde{R}_j \quad (3.7)$$

in $\mathcal{D}(Q)^F$, where α' is the universal map. Therefore (3.7) in $\mathcal{D}(Q)^F$ is induced from (3.6) in $\mathcal{D}(Q)$, which implies (3.5). Similarly for the case of backward tilting.

Via Φ in (1.4), we see that $(\tilde{\mathcal{H}}, \mathcal{H})$ satisfy 3° and the hearts in (3.3) satisfying 1° and 2° as required. \square

We also the corresponding corollary for stability conditions.

Corollary 3.4. *If (Q, \mathbb{S}) one of the Dynkin type in Example 1.1, then ι_Q in (3.1) is a canonical isomorphism. In particular, we have $\mathrm{Stab}^F \mathcal{D}(Q) \cong \mathrm{Stab} \mathcal{D}(\mathbb{S})$.*

Proof. Since \mathbb{S} is of Dynkin type, use the same argument in [11, Appendix A] for example, we have $\mathrm{EG}^\circ \mathcal{D}(\mathbb{S}) = \mathrm{EG} \mathcal{D}(\mathbb{S})$. Thus, $\Phi(\mathrm{EG}^\circ \mathcal{D}(\mathbb{S})) = \Phi(\mathrm{EG} \mathcal{D}(Q))^F$.

For the first claim, we only need to show that ι_Q is surjective, or equivalently, that any heart in $\mathcal{D}(Q)^F$ is the F-stabilization of some heart \mathcal{H} in $\mathcal{D}(Q)$. This follows from Proposition 3.3.

The second claim follows immediately from the derived equivalence (1.4) and the first claim. \square

4. F-STABLE FOR FINITE DIMENSIONAL CATEGORY

We precede to discuss F-stable hearts and stability conditions in $\mathcal{D}(\Gamma Q)$. Notice that formulae (C.4) and (C.5) coincide with twist functor formulae (cf. [9, Remark 7.2]). Hence, similar to the proof of (3.5), we have the following lemma.

Lemma 4.1. *Let the orbit $\mathbf{i} \in \mathbb{S}_0$ consists of vertices $1, \dots, |\mathbf{i}|$ in Q_0 . Then the auto-equivalence*

$$\phi_{\mathbf{i}} = \phi_{T_1} \circ \dots \circ \phi_{T_{|\mathbf{i}|}}$$

preserves F-stable objects in $\mathcal{D}(\Gamma Q)$ and hence induces an auto-equivalence $\phi_{\mathbf{i}}$ on $\mathcal{D}(\Gamma Q)^F$. Moreover, under the derived equivalence (2.1), ϕ_{T_1} corresponds to $\phi_{\mathbf{i}}$.

Denote by $\mathrm{Br} \Gamma Q^F$ the subgroup of $\mathrm{Br} \Gamma Q$ generating by $\{\phi_{\mathbf{i}}\}_{\mathbf{i} \in \mathbb{S}_0}$. Hence, by abuse of notation, we have

$$\Theta(\mathrm{Br} \Gamma \mathbb{S}) = \mathrm{Br} \Gamma Q^F.$$

Now we prove a similar result to Proposition 3.3 for $\mathcal{D}(\Gamma Q)$.

Proposition 4.2. *Any heart in $\Theta(\mathrm{EG}^\circ \mathcal{D}(\Gamma \mathbb{S}))$ is the F-stabilization of some heart in $\mathrm{EG}^\circ \mathcal{D}(\Gamma Q)$.*

Proof. First, by (C.8), for any \mathcal{H}_0 in $\mathrm{EG}^\circ(\mathcal{D}(\Gamma \mathbb{S}), \mathcal{G}_{\mathbb{S}})$, there exists \mathcal{H} in $\mathrm{EG}^\circ(\mathcal{D}(\mathbb{S}), \mathcal{H}_{\mathbb{S}})$ such that $\mathcal{H}_0 = \mathcal{L}_{\mathbb{S}*}(\mathcal{H})$. Then by Proposition 3.3, $\Phi(\mathcal{H}) = \tilde{\mathcal{H}}^F$, for some $\mathcal{H} \in \mathrm{EG}^\circ \mathcal{D}(Q)$. Moreover, by looking at the homology of $\tilde{\mathcal{H}}$ with respect to \mathcal{H}_Q (cf. [9, Lemma 5.4]), \mathcal{H} is actually in $\mathrm{EG}^\circ(\mathcal{D}(Q), \mathcal{H}_Q)$. Further, the quotient map $\Gamma Q \rightarrow \mathbf{k}Q$, which induces the immersion \mathcal{L}_Q , commute with the Frobenius morphisms on $\mathbf{k}Q$ and ΓQ . Therefore we have $\mathcal{L}_{Q*}(\tilde{\mathcal{H}}^F) = \left(\mathcal{L}_{Q*}(\tilde{\mathcal{H}})\right)^F$. Together, we have

$$\Theta(\mathcal{H}_0) = \mathcal{L}_{Q*}(\Phi(\mathcal{H})) = \mathcal{L}_{Q*}(\tilde{\mathcal{H}}^F) = (\mathcal{L}_{Q*}(\tilde{\mathcal{H}}))^F,$$

i.e. $\Theta(\mathcal{H}_0)$ is the F-stabilization of some heart in $\mathrm{EG}^\circ \mathcal{D}(\Gamma Q)$.

By Theorem C.4 and Lemma 4.1 we know that $\Theta(\mathrm{EG}^\circ(\mathcal{D}(\Gamma \mathbb{S}), \mathcal{G}_{\mathbb{S}}))$ is a fundamental domain for $\mathrm{EG}^\circ(\mathcal{D}(\Gamma Q)^F)/\mathrm{Br} \Gamma Q^F$ and $\mathrm{Br} \Gamma Q^F$ preserves F-stabilizing, respectively. Thus all heart in $\mathrm{EG}^\circ(\mathcal{D}(\Gamma \mathbb{S}))$ is the F-stabilization of some heart in $\mathrm{EG}^\circ \mathcal{D}(\Gamma Q)$, as required. \square

Similar to [11, Theorem 4.3], there are principal components

$$\begin{aligned} \mathrm{Stab}_0(\Gamma Q) &= \bigcup_{\mathcal{H} \in \mathrm{EG}^\circ \mathcal{D}(\Gamma Q)} \mathrm{U}(\mathcal{H}), \\ \mathrm{Stab}_0(\Gamma \mathbb{S}) &= \bigcup_{\mathcal{H} \in \mathrm{EG}^\circ \mathcal{D}(\Gamma \mathbb{S})} \mathrm{U}(\mathcal{H}) \end{aligned}$$

in $\text{Stab}(\mathcal{D}(\Gamma Q))$ and $\text{Stab}(\mathcal{D}(\Gamma \mathbb{S}))$, respectively, for any $\mathbb{S} = Q^\sigma$ of Dynkin type in Example 1.1. Denote by $\text{Stab}_0^F(\Gamma Q)$ the subspace in $\text{Stab}_0(\Gamma Q)$ consisting of F-stable stability conditions. As Corollary 3.2, we have an immediate consequence of Proposition 4.2.

Corollary 4.3. *If (Q, \mathbb{S}) is one of the Dynkin type in Example 1.1, then there is a canonical isomorphism*

$$\iota_{\Gamma Q}: \text{Stab}_0^F(\Gamma Q) \cong \text{Stab}_0(\Gamma Q)^F. \quad (4.1)$$

Thus $\text{Stab}_0^F \mathcal{D}(\Gamma Q) \cong \text{Stab}_0 \mathcal{D}(\mathbb{S})$.

Proof. The map $\iota_{\Gamma Q}$ is constructed via the isomorphism between the Grotendieck groups of $K^F(\mathcal{D}(\Gamma Q))$ and $K(\mathcal{D}(\Gamma Q)^F)$ and Theorem 1.3 (cf. commutative diagram (4.2) and (3.2)).

$$\begin{array}{ccc} \text{Stab}_0^F \mathcal{D}(\Gamma Q) & \xrightarrow{\iota_{\Gamma Q}} & \text{Stab}_0 \mathcal{D}(\Gamma Q)^F \\ \downarrow & & \downarrow \\ \text{Hom}(K^F(\mathcal{D}(\Gamma Q)), \mathbb{C}) & \xrightarrow{\cong} & \text{Hom}(K(\mathcal{D}(\Gamma Q)^F), \mathbb{C}). \end{array} \quad (4.2)$$

□

5. NUMERICAL STABILITY CONDITIONS

In this section, we study the space of numerical stability condition of $\mathcal{D}(\Gamma Q)$ via the stability conditions of $\mathcal{D}(\Gamma \mathbb{S})$, for two special Dynkin types, namely (Q, \mathbb{S}) is of type (A_3, B_2) or (D_4, G_2) .

Recall that the *Euler form* on the Grothendieck group $K(\mathcal{G}_Q)$ defined by

$$\chi(\dim M, \dim L) = \sum_i (-1)^i \dim \text{Hom}^i(M, L), \quad (5.1)$$

for any $M, L \in K(\mathcal{G}_Q)$. Since $K(\Gamma Q) := K(\mathcal{D}(\Gamma Q))$ is canonically isomorphic to $K(\mathcal{G}_Q)$, χ is also a bilinear form on $K(\Gamma Q)$. A numerical stability condition on $\mathcal{D}(\Gamma Q)$ is a stability condition (\mathcal{H}, Z) such that the central charge $Z : K(\Gamma Q) \rightarrow \mathbb{C}$ factors through the numerical Grothendieck group $K(\Gamma Q)/Z_\chi(\Gamma Q)$, where

$$Z_\chi(\Gamma Q) = \{X \in K(\Gamma Q) \mid \chi(X, Y) = 0, \forall Y \in K(\Gamma Q)\}.$$

Denote by $\text{Stab}^N(\Gamma Q)$ the space of numerical stability conditions that are in $\text{Stab}_0(\Gamma Q)$; denote by $\text{Stab}_0^N(\Gamma Q)$ its principal component, that is, the connected component contains the numerical stability condition with heart \mathcal{G}_Q .

Now we prove the main theorem.

Theorem 5.1. *For (Q, \mathbb{S}) is of type (A_3, B_2) and (D_4, G_2) as in Example 1.1, $\text{Stab}^N(\Gamma Q)$ consists of $\text{Br } \Gamma Q / \text{Br } \Gamma \mathbb{S}$ many (connected) components, each of which is isomorphic to*

$$\text{Stab}_0^N(\Gamma Q) = \text{Stab}_0^F \mathcal{D}(\Gamma Q) \cong \text{Stab}_0(\Gamma \mathbb{S}).$$

Proof. We only deal with the case (Q, \mathbb{S}) is of type (A_3, B_2) , while the other case is similar. Without loss of generality, suppose that the labeling and the orientations of (Q, \mathbb{S}) are

$$Q: \begin{array}{c} & 1 \\ 2 & \nearrow \searrow \\ & 3 \end{array} \quad \mathbb{S}: \quad \begin{array}{c} 2 \rightleftharpoons 1 \end{array} \quad (5.2)$$

Recall that $\text{Sim } \mathcal{G}_Q = \{T_1, T_2, T_3\}$. By a direct calculation, we know that

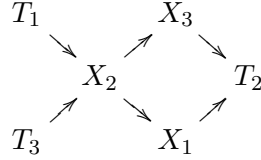
- $F(T_1) = T_3$, $F(T_2) = T_2$, $F(T_3) = T_1$ and hence $F^2 = \text{id}$. Thus a stability condition (\mathcal{H}, Z) is F-stable if and only if $Z(T_1) = Z(T_3)$ and \mathcal{H} is F-stable.
- $Z_\chi(\Gamma Q)$ is generated by $[T_1] - [T_3]$. Thus a stability condition (\mathcal{H}, Z) is numerical if and only if $Z(T_1) = Z(T_3)$.

Clearly, a F-stable stability condition is numerical.

Next, we investigate stability conditions in

$$\mathcal{S} := \bigcup_{\mathcal{H} \in \text{EG}^\circ(\mathcal{D}(\Gamma Q), \mathcal{G}_Q)} \text{U}(\mathcal{H}), \quad (5.3)$$

where $\text{EG}^\circ(\mathcal{D}(\Gamma Q), \mathcal{G}_Q)$ is defined as in Appendix C. The Auslander-Reiten quiver of \mathcal{G}_Q is as following



$\text{EG}^\circ(\mathcal{D}(\Gamma Q), \mathcal{G}_Q)$ is shown in Figure 1, where we denote each heart by a complete set of simples and $?^1 = ?[1]$. Note that the F-stable ones are underlined.

Let \mathcal{H} be a non-F-stable heart in Figure 1 and we claim that all stability conditions in $\text{U}(\mathcal{H})$ are not numerical. To see this, take the top heart in Figure 1 for example. Then $Z(T_1)$ and $Z(T_3[1])$ are in the same upper half plane H as in (1.2). Thus $Z(T_1) = Z(T_3)$ never holds, which implies the claim.

Let \mathcal{H} be an F-stable heart in Figure 1. Then if a stability condition with heart \mathcal{H} is numerical if and only if it is F-stable, i.e.

$$\text{U}(\mathcal{H}) \cap \text{Stab}^N \mathcal{D}(\Gamma Q) = \text{U}(\mathcal{H}) \cap \text{Stab}_0^F \mathcal{D}(\Gamma Q). \quad (5.4)$$

Therefore, we have

$$\mathcal{S}_* := \mathcal{S} \cap \text{Stab}^N \mathcal{D}(\Gamma Q) = \mathcal{S} \cap \text{Stab}_0^F \mathcal{D}(\Gamma Q).$$

Notice that $\text{Br } \Gamma Q^F$ preserves F-stable stability conditions and all auto-equivalence $\text{Aut } \mathcal{D}(\Gamma Q)$ preserves numerical stability conditions. Then by (C.9) we have

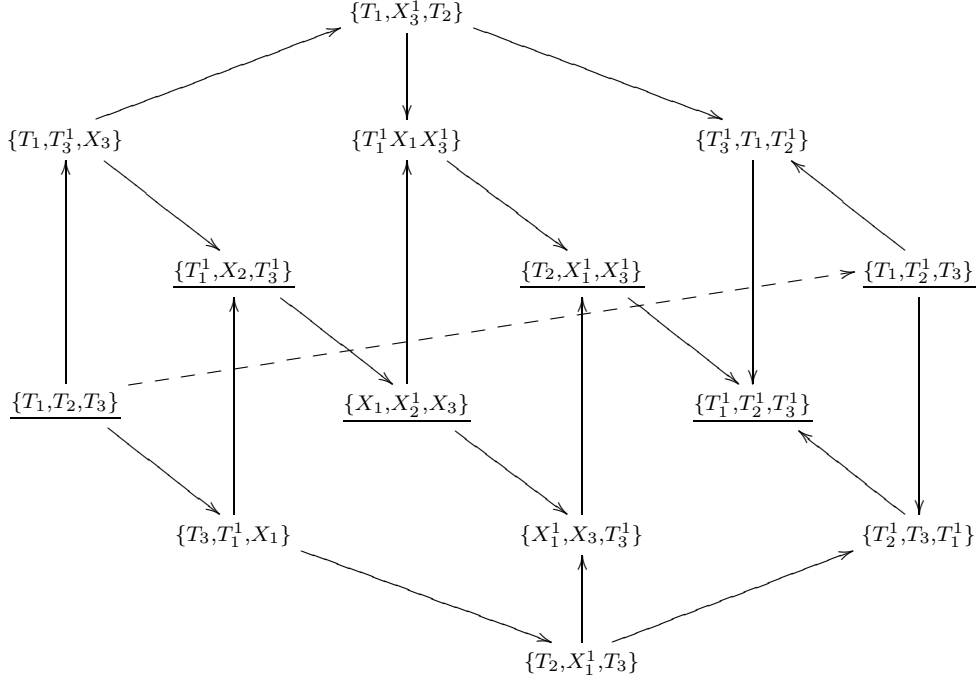
$$\text{Stab}^N \mathcal{D}(\Gamma Q) = \text{Br } \Gamma Q \cdot \mathcal{S}_*.$$

Similarly, by (C.10) and (2.1) we have

$$\text{EG}^\circ(\mathcal{D}(\Gamma Q)^F, \mathcal{G}_Q^F) \cong \text{EG}^\circ \mathcal{D}(\Gamma Q)^F / \text{Br } \Gamma Q^F$$

and hence

$$\text{Stab}_0^F \mathcal{D}(\Gamma Q) = \text{Br } \Gamma Q^F \cdot \mathcal{S}_*.$$

FIGURE 1. The exchange graph $\text{EG}^\circ(\Gamma Q, \mathcal{G}_Q)$ for Q of type A_3

Thus $\text{Stab}^N \mathcal{D}(\Gamma Q)$ is the union of $\text{Br } \Gamma Q / \text{Br } \Gamma Q^F$ many copy of $\text{Stab}_0^F \mathcal{D}(\Gamma Q)$.

To finish, we assert that the closure of $\text{Stab}_0^F \mathcal{D}(\Gamma Q)$, which is taken inside $\text{Stab}_0 \mathcal{D}(\Gamma Q)$, is disjoint with

$$C_0 := \text{Stab}^N \mathcal{D}(\Gamma Q) - \text{Stab}_0^F \mathcal{D}(\Gamma Q).$$

If so, $\text{Stab}_0^F \mathcal{D}(\Gamma Q)$ and C_0 are then both closed, noticing that C_0 is the union of many copy of $\text{Stab}_0^F \mathcal{D}(\Gamma Q)$. This will imply that $\text{Stab}_0^F \mathcal{D}(\Gamma Q)$ is a connected component of $\text{Stab}^N(\Gamma Q)$ and hence the theorem follows.

The rest of the proof is devoted to prove the assertion. Let $\text{EG}^F = \text{EG}^\circ \mathcal{D}(\Gamma Q)^F$ and $U^F(\mathcal{H}) = U(\mathcal{H}) \cap \text{Stab}_0^F \mathcal{D}(\Gamma Q)$ for any $\mathcal{H} \in \text{EG}^F$. First, we have

$$\overline{\text{Stab}_0^F \mathcal{D}(\Gamma Q)} = \bigcup_{\mathcal{H} \in \text{EG}^F} \overline{U^F(\mathcal{H})}$$

and thus we only need to show that, for any $\mathcal{H} \in \text{EG}^F$,

$$\overline{U^F(\mathcal{H})} \cap C_0 = \emptyset. \quad (5.5)$$

Without lose of generality, take the F-stable heart $\mathcal{H} = \mathcal{G}_Q[1]$. By formula [11, (3.1)], we have

$$\overline{U^F(\mathcal{G}_Q[1])} \subset \overline{U(\mathcal{G}_Q[1])} \subset \bigcup_{\mathcal{H} \in \text{EG}^\circ(\mathcal{D}(\Gamma Q), \mathcal{G}_Q)} U(\mathcal{H}). \quad (5.6)$$

We also have

$$C_0 \subset \bigcup_{\mathcal{H} \in \text{Br } \Gamma Q \cdot \text{EG}^F - \text{EG}^F} U(\mathcal{H}). \quad (5.7)$$

But (C.9) implies that

$$\mathrm{Br} \Gamma Q \cdot \mathrm{EG}^F \cap \mathrm{EG}^\circ(\mathcal{D}(\Gamma Q), \mathcal{G}_Q) = \mathrm{EG}^F \cap \mathrm{EG}^\circ(\mathcal{D}(\Gamma Q), \mathcal{G}_Q) \subset \mathrm{EG}^F$$

and hence

$$(\mathrm{Br} \Gamma Q \cdot \mathrm{EG}^F - \mathrm{EG}^F) \cap \mathrm{EG}^\circ(\mathcal{D}(\Gamma Q), \mathcal{G}_Q) = \emptyset. \quad (5.8)$$

Combine (5.6), (5.7) and (5.8), we have (5.5) for $\mathcal{H} = \mathcal{G}_Q[1]$ as required. \square

Remark 5.2. If (Q, \mathbb{S}) is of type (A_3, B_2) as in (5.2), then $\mathrm{Br} \Gamma \mathbb{S}$ satisfies the B_2 -braid relation, i.e.

$$(\mathbf{1} \circ \mathbf{2})^2 = \mathbf{1} \circ \mathbf{2} \circ \mathbf{1} \circ \mathbf{2} = \mathbf{2} \circ \mathbf{1} \circ \mathbf{2} \circ \mathbf{1} = (\mathbf{2} \circ \mathbf{1})^2,$$

where \mathbf{i} represent the twist functor of $T_{\mathbf{i}}$ in $\mathrm{Br} \Gamma \mathbb{S}$. This follows by Lemma 4.1 and a direct calculation

$$\begin{aligned} (1 \circ 3) \circ 2 \circ (\underline{1 \circ 3}) \circ 2 &= 1 \circ \underline{3 \circ 2 \circ 3} \circ 1 \circ 2 \\ &= 1 \circ 2 \circ 3 \circ \underline{2 \circ 1 \circ 2} &= 1 \circ 2 \circ \underline{3 \circ 1} \circ 2 \circ 1 \\ &= \underline{1 \circ 2 \circ 1} \circ 3 \circ 2 \circ 1 &= 2 \circ 1 \circ \underline{2 \circ 3 \circ 2} \circ 1 \\ &= 2 \circ 1 \circ 3 \circ 2 \circ \underline{3 \circ 1} &= 2 \circ (1 \circ 3) \circ 2 \circ (1 \circ 3) \end{aligned}$$

where i represent the twist functor of T_i in $\mathrm{Br} \Gamma Q \cong \mathrm{Br}_4$. Thus

$$\mathrm{Br}_{B_2} \cong \mathrm{Br} \Gamma \mathbb{S} \subset \mathrm{Br} \Gamma Q \cong \mathrm{Br}_3.$$

Similarly, If (Q, \mathbb{S}) is of type (D_4, G_2) , then $\mathrm{Br} \Gamma \mathbb{S}$ satisfies the G_2 -braid relation, i.e.

$$(\mathbf{1} \circ \mathbf{2})^3 = (\mathbf{2} \circ \mathbf{1})^3$$

and we have

$$\mathrm{Br}_{G_2} \cong \mathrm{Br} \Gamma \mathbb{S} \subset \mathrm{Br} \Gamma Q \cong \mathrm{Br}_{D_4}$$

(note that we need the faithfulness [10] of $\mathrm{Br} \Gamma Q \cong \mathrm{Br}_{D_4}$ in this case).

APPENDIX A. FOLDING

Recall that an \mathbb{F}_q -species $\mathbb{S} = (\mathbb{S}_0, L_{\mathbf{i}}, X_{\mathbf{a}})$ consists of the following data:

- A quiver $\mathbb{S} = (\mathbb{S}_0, \mathbb{S}_1)$.
- A division ring (over \mathbb{F}_q) $L_{\mathbf{i}}$ for each vertex $\mathbf{i} \in \mathbb{S}_0$.
- A $L_{\mathbf{i}}-L_{\mathbf{j}}$ -bimodule $X_{\mathbf{a}}$ for each arrow $\mathbf{a}: \mathbf{i} \rightarrow \mathbf{j}$ in \mathbb{S}_1 .

Following [2, Section 2], we collect facts about folding. Let Q be an acyclic quiver and σ be an automorphism of Q , that is, a permutation on Q_0 satisfying the compatibility conditions $h(\sigma(a)) = \sigma(h(a))$ and $t(\sigma(a)) = \sigma(t(a))$ for any $a \in Q_1$, where h, t are the head and tail functions on arrows. We assume that σ is admissible, that is, there are no arrows connecting vertices in the same orbit of σ in Q_0 . Then there is a Frobenius morphism $F = F_Q^\sigma(q)$ on $\mathbf{k}Q$ given by

$$F\left(\sum_s k_s p_s\right) = \varsigma(k_s) \sigma(p_s), \quad (\text{A.1})$$

where \sum_s is a \mathbf{k} -linear combination of paths in $\mathbf{k}Q$. Define the *associate* \mathbb{F}_q -species $\mathbb{S} = Q^\sigma$ of (Q, σ) as follows.

- The quiver \mathbb{S} is the σ -orbit of Q , i.e. $\mathbb{S}_0 = Q_0/\sigma$ and $\mathbb{S}_1 = Q_1/\sigma$.

- For $\mathbf{i} \in \mathbb{S}_0$, denote by $|\mathbf{i}|$ the number of vertices in the σ -orbit \mathbf{i} and fix $i_0 \in \mathbf{i}$. Consider the F -stable subspaces of $\mathbf{k}Q$

$$(\mathbf{k}Q)_{\mathbf{i}} = \bigoplus_{i \in \mathbf{i}} \mathbf{k}e_i = \bigoplus_{s=1}^{|\mathbf{i}|} \mathbf{k}e_{\sigma^s(i_0)},$$

and let

$$L_{\mathbf{i}} = (\mathbf{k}Q)_{\mathbf{i}}^F = \left\{ \sum_{s=1}^{|\mathbf{i}|} \varsigma^s(x) e_{\sigma^s(i_0)} \mid x \in \mathbf{k}^{\sigma^{|\mathbf{i}|}} \right\},$$

where V^F is the set of F -stable objects in V .

- For $\mathbf{a} \in \mathbb{S}_1$, similarly define $|\mathbf{a}|$ and fix $a_0 \in \mathbf{a}$. Consider the F -stable subspaces of $\mathbf{k}Q$

$$(\mathbf{k}Q)_{\mathbf{a}} = \bigoplus_{a \in \mathbf{a}} \mathbf{k}e_a = \bigoplus_{s=1}^{|\mathbf{a}|} \mathbf{k}\sigma^s(a_0).$$

and let

$$X_{\mathbf{a}} = (\mathbf{k}Q)_{\mathbf{a}}^F = \left\{ \sum_{s=1}^{|\mathbf{a}|} \varsigma^s(x) \sigma^s(a_0) \mid x \in \mathbf{k}^{\sigma^{|\mathbf{a}|}} \right\}.$$

Notice that $X_{\mathbf{a}}$ is a $L_{\mathbf{i}}\text{-}L_{\mathbf{j}}$ -bimodule, by the induced algebraic structure, for $\mathbf{a}: \mathbf{i} \rightarrow \mathbf{j}$ in \mathbb{S} .

APPENDIX B. CALABI-YAU CATEGORIES

Let \mathbf{k}' be a field (not necessarily algebraically closed). Recall that a \mathbf{k}' -linear triangulated category \mathcal{D} is called *Calabi-Yau- N* if, for any objects L, M in \mathcal{D} we have a natural pairing

$$\mathrm{Hom}^i(L, M) \otimes_{\mathrm{End}(M)} \mathrm{Hom}^{N-i}(M, L) \xrightarrow{\cong} \mathrm{Hom}^N(L, L). \quad (\text{B.1})$$

Further, an object T is *N -spherical \mathbf{k}' -object* if

$$\mathrm{Hom}^\bullet(T, T) = \mathbf{k}' \oplus \mathbf{k}'[-N].$$

Note that a N -spherical \mathbf{k}' -object T in a triangulated category \mathcal{D} induces the *twist functor*

$$\phi_T(X) = \mathrm{Cone}(X \rightarrow \mathrm{Hom}^\bullet(X, T)^\vee \otimes_{\mathbf{k}'} T)[-1]$$

with inverse

$$\phi_T^{-1}(X) = \mathrm{Cone}(\mathrm{Hom}^\bullet(T, X) \otimes_{\mathbf{k}'} T \rightarrow X)$$

in $\mathrm{Aut} \mathcal{D}$. Here, the dual of a graded \mathbf{k}' -vector space $V = \bigoplus_{i \in \mathbb{Z}} V_i[i]$ is

$$V_{\mathbf{k}'}^\vee = \bigoplus_{i \in \mathbb{Z}} (V_i)_{\mathbf{k}'}^*[-i]$$

where V_i is an ungraded \mathbf{k}' -vector space and $(V_i)_{\mathbf{k}'}^*$ is its usual dual (with respect to \mathbf{k}').

Denote by ΓQ the *Ginzburg (dg) algebra* (of degree 3) associated to Q , which is constructed as follows ([8, Section 7.2]):

- Let \overline{Q} be the graded quiver whose vertex set is Q_0 and whose arrows are: the arrows in Q with degree 0; an arrow $a^* : j \rightarrow i$ with degree -1 for each arrow $a : i \rightarrow j$ in Q ; a loop $e^* : i \rightarrow i$ with degree -2 for each vertex e in Q .
- The underlying graded algebra of ΓQ is the completion of the graded path algebra $\mathbf{k}\overline{Q}$ in the category of graded vector spaces with respect to the ideal generated by the arrow of \overline{Q} .
- The differential of ΓQ is the unique continuous linear endomorphism homogeneous of degree 1 which satisfies the Leibniz rule and takes the following values on the arrow of \overline{Q}

$$d \sum_{e \in Q_0} e^* = \sum_{a \in Q_1} [a, a^*].$$

Write $\mathcal{D}(\Gamma Q)$ for $\mathcal{D}_{fd}(\text{mod } \Gamma Q)$, the *finite dimensional derived category* of ΓQ (cf. [8, Section 7.3]). By [7] (cf. [12]), we know the following

- $\mathcal{D}(\Gamma Q)$ is (\mathbf{k}' -linear and) Calabi-Yau-3.
- $\mathcal{D}(\Gamma Q)$ admits a standard heart \mathcal{G}_Q , generated by the simple ΓQ -modules T_i .
- T_i is a 3-spherical \mathbf{k}' -object, for any $i \in Q_0$.

Denote by $\text{Br } \Gamma Q$ the *Seidel-Thomas braid group*, that is, the subgroup of $\text{Aut } \mathcal{D}(\Gamma Q)$ generating by $\{\phi_T\}_{T \in \text{Sim } \mathcal{G}_Q}$.

Similarly, we have the Ginzburg algebra of a \mathbb{F}_q specie $\mathbb{S} = (\mathbb{S}, L_{\mathbf{i}}, X_{\mathbf{a}})$, constructed as follows

- Let $\overline{\mathbb{S}}$ be the graded species whose vertex set is \mathbb{S}_0 (associated with the same division rings) and whose arrows are: the arrows in \mathbb{S} with degree 0 and the same bimodules; an arrow $\mathbf{a}^* : \mathbf{j} \rightarrow \mathbf{i}$ with degree -1 and a $L_{\mathbf{j}}\text{-}L_{\mathbf{i}}$ -bimodule $(X_{\mathbf{a}})_{\mathbb{F}_q}^*$, for each arrow $\mathbf{a} : \mathbf{i} \rightarrow \mathbf{j}$ in \mathbb{S}_1 ; a loop $\mathbf{i}^* : \mathbf{i} \rightarrow \mathbf{i}$ with degree -2 with a $L_{\mathbf{i}}\text{-}L_{\mathbf{i}}$ -bimodule $(L_{\mathbf{i}})_{\mathbb{F}_q}^*$ for each vertex \mathbf{i} in \mathbb{S} .
- The underlying graded algebra of $\Gamma \mathbb{S}$ is the completion of the graded path algebra $\mathbf{k}\overline{\mathbb{S}}$ in the category of graded vector spaces with respect to the ideal generated by the arrow of $\overline{\mathbb{S}}$.
- The differential of $\Gamma \mathbb{S}$ is the unique continuous linear endomorphism homogeneous of degree 1 which satisfies the Leibniz rule and takes the following values on the arrow of $\overline{\mathbb{S}}$

$$d \sum_{\mathbf{i} \in \mathbb{S}_0} \mathbf{i}^* = \sum_{\mathbf{a} \in \mathbb{S}_1} [\mathbf{a}, \mathbf{a}^*].$$

Write $\mathcal{D}(\Gamma \mathbb{S})$ for $\mathcal{D}_{fd}(\text{mod } \Gamma \mathbb{S})$, the derived category of finite dimensional dg modules for $\Gamma \mathbb{S}$. As above, we have the following:

- $\mathcal{D}(\Gamma \mathbb{S})$ is (\mathbb{F}_q -linear and) Calabi-Yau-3.
- $\mathcal{D}(\Gamma \mathbb{S})$ admits a standard heart $\mathcal{G}_{\mathbb{S}}$, generated by the simple $\Gamma \mathbb{S}$ -modules $T_{\mathbf{i}}$.
- $T_{\mathbf{i}}$ is 3-spherical $L_{\mathbf{i}}$ -object, for any $\mathbf{i} \in \mathbb{S}_0$.

Denote by $\text{Br } \Gamma \mathbb{S}$ the *Seidel-Thomas braid group*, that is, the subgroup of $\text{Aut } \mathcal{D}(\Gamma \mathbb{S})$ generated by $\{\phi_T\}_{T \in \text{Sim } \mathcal{G}_{\mathbb{S}}}$.

APPENDIX C. IMPORTED RESULTS

In this section, we collect results, which are straightforward generalizations of original ones. The difference is that we need to consider fields (i.e. \mathbb{F}_q) which are not algebraic closed. One major consequence is that the tensor need to be carefully dealt.

The first two results concern (simple) tilting.

Lemma C.1. [9, Lemma 3.5] *Let S be a rigid simple object with $E = \text{End}(S)$ in a Hom-finite abelian \mathbb{F}_q -category \mathcal{C} . Then \mathcal{C} admits a torsion pair $\langle \mathcal{F}, \mathcal{T} \rangle$ such that $\mathcal{F} = \langle S \rangle$. More precisely, for any $M \in \mathcal{H}$, in the corresponding short exact sequence*

$$0 \rightarrow M^{\mathcal{T}} \rightarrow M \rightarrow M^{\mathcal{F}} \rightarrow 0 \quad (\text{C.1})$$

we have $M^{\mathcal{F}} = \text{Hom}(M, S)^ \otimes_E S$. Similarly, there is also a torsion pair with the torsion part $\mathcal{T} = \langle S \rangle$, obtained by setting $M^{\mathcal{T}} = \text{Hom}(S, M) \otimes_E S$.*

Proposition C.2. [9, Proposition 5.2] *In an \mathbb{F}_q -linear triangulated category \mathcal{D} , let S be a rigid simple in a finite heart \mathcal{H} . Then after a forward or backward simple tilt the new simples are*

$$\text{Sim } \mathcal{H}_S^{\sharp} = \{ \psi_S^{\sharp}(X) \mid X \in \text{Sim } \mathcal{H}, X \neq S \} \cup \{ S[1] \}, \quad (\text{C.2})$$

$$\text{Sim } \mathcal{H}_S^{\flat} = \{ \psi_S^{\flat}(X) \mid X \in \text{Sim } \mathcal{H}, X \neq S \} \cup \{ S[-1] \}, \quad (\text{C.3})$$

where

$$\psi_S^{\sharp}(X) = \text{Cone}(X \rightarrow \text{Ext}^1(X, S)^* \otimes_E S[1])[-1], \quad (\text{C.4})$$

$$\psi_S^{\flat}(X) = \text{Cone}(\text{Ext}^1(S, X) \otimes_E S[-1] \rightarrow X) \quad (\text{C.5})$$

and $E = \text{End}(S)$. Thus \mathcal{H}_S^{\sharp} and \mathcal{H}_S^{\flat} are also finite.

With the notation in Section 1, we have the following.

Proposition C.3. [6] *There is an exact and faithful functor (called L-immersion, cf. [9, Section 7.3])*

$$\mathcal{L}_Q: \mathcal{D}(Q) \rightarrow \mathcal{D}(\Gamma Q)$$

such that, for any pair of objects $(\widehat{S}, \widehat{X})$ in $\mathcal{D}(Q)$, there is a short exact sequence

$$0 \rightarrow \text{Hom}^{\bullet}(\widehat{S}, \widehat{X}) \xrightarrow{\mathcal{L}_Q} \text{Hom}^{\bullet}(\mathcal{L}_Q(\widehat{S}), \mathcal{L}_Q(\widehat{X})) \xrightarrow{\mathcal{L}_Q^{\dagger}} \text{Hom}^{\bullet}(\widehat{X}, \widehat{S})_{\mathbf{k}}^{\vee}[N] \rightarrow 0, \quad (\text{C.6})$$

Moreover, we have $\mathcal{L}_Q(\mathcal{H}_Q) = \mathcal{G}_Q$ and $\mathcal{L}_Q(S_i) = T_i$, for any $i \in Q_0$. Similarly, we have an L-immersion

$$\mathcal{L}_S: \mathcal{D}(\mathbb{S}) \rightarrow \mathcal{D}(\Gamma \mathbb{S})$$

satisfying the corresponding (C.6).

For any $\mathcal{H}_0 \in \text{EG } \mathcal{D}$, let the interval $\text{EG}(\mathcal{D}, \mathcal{H}_0)$ be the full subgraph of $\text{EG } \mathcal{D}$ induced by the interval $\{ \mathcal{H} \mid \mathcal{H}_0 \leq \mathcal{H} \leq \mathcal{H}_0[1] \}$ and $\text{EG}^{\circ}(\mathcal{D}, \mathcal{H})$ be its principal component, that is, the connected component consisting of hearts that can be iterated simple tilted from \mathcal{H}_0 . We have the following.

Theorem C.4. [9, Theorem 8.1 and 8.5] *There are isomorphisms (as graph)*

$$\mathcal{L}_{Q*}: \mathrm{EG}^\circ(\mathcal{D}(Q), \mathcal{H}_Q) \cong \mathrm{EG}^\circ(\mathcal{D}(\Gamma Q), \mathcal{G}_Q), \quad (\text{C.7})$$

$$\mathcal{L}_{\mathbb{S}*}: \mathrm{EG}^\circ(\mathcal{D}(\mathbb{S}), \mathcal{H}_{\mathbb{S}}) \cong \mathrm{EG}^\circ(\mathcal{D}(\Gamma \mathbb{S}), \mathcal{G}_{\mathbb{S}}) \quad (\text{C.8})$$

and isomorphisms (as vertex set)

$$\mathrm{EG}^\circ(\mathcal{D}(\Gamma Q), \mathcal{G}_Q) \cong \mathrm{EG}^\circ \mathcal{D}(\Gamma Q) / \mathrm{Br} \Gamma Q, \quad (\text{C.9})$$

$$\mathrm{EG}^\circ(\mathcal{D}(\Gamma \mathbb{S}), \mathcal{G}_{\mathbb{S}}) \cong \mathrm{EG}^\circ \mathcal{D}(\Gamma \mathbb{S}) / \mathrm{Br} \Gamma \mathbb{S}. \quad (\text{C.10})$$

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